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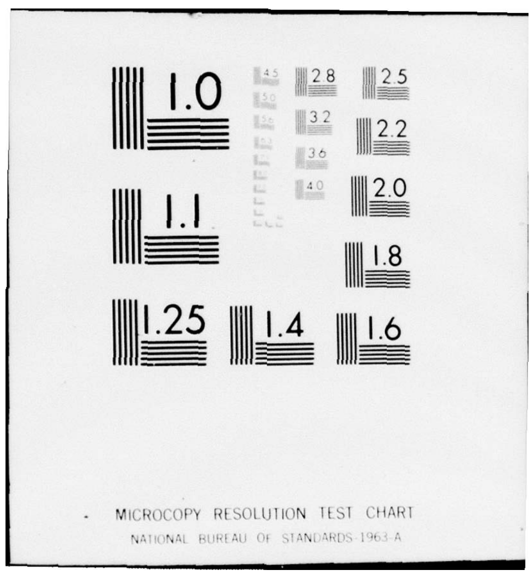
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MULTI-VALUED STATE COMPONENT RELIABILITY SYSTEMS

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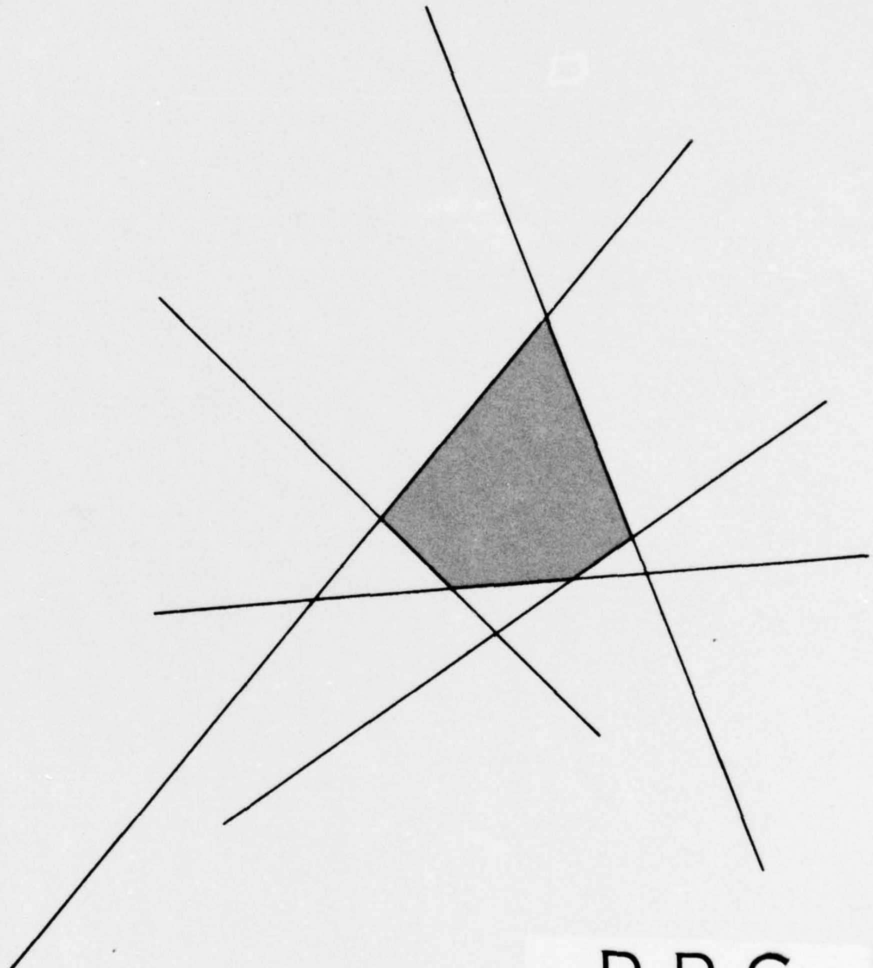
by
SHELDON M. ROSS



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MULTI-VALUED STATE COMPONENT RELIABILITY SYSTEMS

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JUNE 1977 ORC 77-18

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ABSTRACT

Consider a reliability system that is composed of n components each of which is operating at some performance level. ~~We~~ And suppose that there exists a nondecreasing function ϕ, called the structure function, such that $\phi(x_1, \ldots, x_n)$ denotes the performance level of the system when the $i^{\underline{th}}$ component's performance level is x_i, $i = 1, \ldots, n$.

Whereas almost all previous work assumed that both x_i and $\phi(x_1, \ldots, x_n)$ were binary variables we shall allow both to be arbitrary nonnegative numbers and we extend many of the important results of the usual binary model to this more general framework. In particular ~~we obtain~~ a fundamental inequality is obtained for $E[\phi(X_1, \ldots, X_n)]$ when ϕ is binary which can, among other things, be used to generate a host of inequalities concerning IFRA distributions including, as a special case, the IFRA convolution theorem. In Section 2 we define the concept of an IFRA stochastic process and prove the analog of the IFRA closure theorem; and in Section 3 we do the same for NBU stochastic processes. In the final section we present some applications to stochastic networks.

MULTI-VALUED STATE COMPONENT RELIABILITY SYSTEMS

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0. INTRODUCTION AND SUMMARY

Consider a reliability system that is composed of n components each of which is operating at some performance level. We suppose that there exists a nondecreasing function ϕ, called the structure function, such that $\phi(x_1, \ldots, x_n)$ denotes the performance level of the system when the $i^{\underline{th}}$ component's performance level is x_i, $i = 1, \ldots, n$.

Whereas almost all previous work has assumed that both x_i and $\phi(x_i, \ldots, x_n)$ were binary variables we shall allow both to be arbitrary nonnegative numbers. In the next few sections we extend many of the important results of the usual binary model to this more general framework. In particular we obtain, in Section 1, a fundamental inequality for $E[\phi(X_1, \ldots, X_n)]$ when ϕ is binary which can, among other things, be used to generate a host of inequalities concerning IFRA distributions including, as a special case, the IFRA convolution theorem. In Section 2 we define the concept of an IFRA stochastic process and prove the analog of the IFRA closure theorem; and in Section 3 we do the same for NBU stochastic processes. In the final Section we present some applications to stochastic networks.

1. THE STRUCTURE FUNCTION

Suppose now the performance level of component i is a random variable X_i having distribution $\bar{F}_i$ where $\bar{F}_i(x) = P\{X_i > x\}$, and suppose that the X_i are independent. We define the function $r(F_1, \ldots, F_n)$ by

$$r(\bar{F}_1, \ldots, \bar{F}_n) = E[\phi(X_1, \ldots, X_n)]$$

and call r the reliability function of the system. It immediately follows from the monotonicity of ϕ that

Proposition 1:

If $\bar{F}_i$ and $\bar{G}_i$ are distributions such that $\bar{F}_i(x) \geq \bar{G}_i(x)$ for all x then

$$r(\bar{F}_1, \ldots, \bar{F}_n) \geq r(\bar{G}_1, \ldots, \bar{G}_n) .$$

We shall need the following lemma which is a slight variation of a lemma used by Block and Savits [3] to prove the IFRA convolution theorem.

Lemma 1:

Let $r(s)$ be a nonnegative nondecreasing function of s, $s \geq 0$ and let $\bar{G}$ be a distribution function with $\bar{G}(0) = 1$. Then for $0 < \alpha \leq 1$,

$$\int_0^\infty (r(s))^\alpha d(1 - \bar{G}^\alpha(s)) \geq \left[\int_0^\infty r(s) d(1 - \bar{G}(s))\right]^\alpha .$$

Proof:

Lemma 4.1 (p. 217) of Ross [5] (also given as Lemma 2.3 (p. 84) of

Barlow and Proschan [1] generalizes to give that for

$$0 \le x_1 \le x_2 \le \cdots \le x_n \ , \ y_i \ge 0 \ , \ \sum_i^n y_i > 0$$

$$\left(\sum_1^n r(x_i)y_i\right)^\alpha \le \sum_1^n (r(x_i))^\alpha \left[\left(\sum_{k=i}^n y_k\right)^\alpha - \left(\sum_{k=i+1}^n y_k\right)^\alpha\right] .$$

From this the conclusion follows from a standard limiting argument (as in [3]).■

The following theorem is of fundamental importance.

Theorem 1:

If ϕ is a binary function then

$$(*) \qquad r\left(\bar{F}_1^\alpha, \ldots, \bar{F}_n^\alpha\right) \ge [r(\bar{F}_1, \ldots, \bar{F}_n)]^\alpha$$

for all $0 \le \alpha \le 1$.

Proof:

The proof is by induction of n . When $n = 1$ it follows from the monotonicity of ϕ that it must be of the form $\phi(x) = \begin{cases} 1 & x > c \\ 0 \end{cases}$ for some c . Hence $E[\phi(X_1)] = \bar{F}_1(c)$ and so both sides of the inequality (*) are equal. So assume (*) for all binary structures of $n - 1$ components, and consider the n component case. Conditioning on X_n yields

$$(1) \qquad r\left(\bar{F}_1^\alpha, \ldots, \bar{F}_n^\alpha\right) = \int r_s\left(\bar{F}_1^\alpha, \ldots, \bar{F}_{n-1}^\alpha\right) d\left(1 - \bar{F}_n^\alpha(s)\right)$$

where

$$r_s\left(\bar{F}_1^\alpha, \ldots, \bar{F}_{n-1}^\alpha\right) = E[\phi(X_1, \ldots, X_{n-1}, s)]$$

with X_i having distribution $\bar{F}_i$. By the induction hypothesis we see that

$$r_s\left(\bar{F}_1^\alpha, \ldots, \bar{F}_{n-1}^\alpha\right) \geq [r_s(\bar{F}_1, \ldots, \bar{F}_{n-1})]^\alpha$$

and so from (1),

$$r\left(\bar{F}_1^\alpha, \ldots, \bar{F}_n^\alpha\right) \geq \int (r_s[\bar{F}_1, \ldots, \bar{F}_{n-1})]^\alpha d\left(1 - \bar{F}_n^\alpha(s)\right) .$$

As it follows from the monotonicity of ϕ that r_s is nondecreasing in s we can apply Lemma 1 to the above to obtain that

$$r\left(\bar{F}_1^\alpha, \ldots, F_n^\alpha\right) \geq \left[\int_0^\infty r_s(\bar{F}_1, \ldots, \bar{F}_{n-1}) d(1 - \bar{F}_n(s))\right]^\alpha$$

$$= (r(\bar{F}_1, \ldots, F_n))^\alpha . \blacksquare$$

Definition:

The distribution function $\bar{F}$, with $\bar{F}(0) = 1$, is said to be an IFRA distribution if

$$\bar{F}(\alpha x) \geq \bar{F}^\alpha(x) \quad \text{for all} \quad 0 \leq \alpha < 1 , \; x \geq 0 .$$

Corollary 1:

If $X_1, \ldots, X_n$ are independent random variables, each having an IFRA distribution, then for all nondecreasing binary functions ϕ,

$$E\left[\phi\left(\frac{X_1}{\alpha}, \frac{X_2}{\alpha}, \ldots, \frac{X_n}{\alpha}\right)\right]$$

$$\geq (E[\phi(X_1, \ldots, X_n)])^\alpha \quad \text{for} \quad 0 \leq \alpha < 1 .$$

Proof:

If X_i is IFRA with distribution F_i, then $P\left\{\frac{X_i}{\alpha} > x\right\} \geq \bar{F}_i^\alpha(x)$

and so from Proposition 1

$$E\left[\phi\left(\frac{X_1}{\alpha}, \ldots, \frac{X_n}{\alpha}\right)\right] \geq r\left(\bar{F}_1, \ldots, \bar{F}_n^{\alpha}\right)$$

and the result follows from Theorem 1.■

The above Corollary provides a host of inequalities concerning IFRA random variables. For instance we have

Corollary 2:

If $X_1, \ldots, X_n$ are independent IFRA random variables then

(a) $\sum_1^n X_i$ is IFRA.

(b) $P\left\{\prod_1^n X_i > a\alpha^n\right\} \geq \left(P\left\{\prod_1^n X_i > a\right\}\right)^{\alpha} \quad 0 \leq \alpha \leq 1$.

Proof:

Part 1 follows from Corollary 1 by using the function

$$\phi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \sum_i^n X_i > a \\ 0 & \text{otherwise} \end{cases}$$

to obtain

$$P\left\{\sum_i^n X_i > \alpha a\right\} \geq \left(P\left\{\sum X_i > a\right\}\right)^{\alpha}$$

similarly (b) follows by using

$$\phi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \Pi X_i \geq a \\ 0 & \text{otherwise.} \end{cases}$$ ■

2. THE GENERALIZED IFRA CLOSURE THEOREM

In this section we suppose that the component performance levels vary with time and we let $X_i(t)$ denote the level of component i at time t. Thus, for instance, $\phi(\underline{X}(t)) = \phi(X_1(t), \ldots, X_n(t))$ denotes the systems performance level at time t.

Definition:

The real-valued stochastic process $\{X(t), t \geq 0\}$ is said to be an IFRA process if T_a is an IFRA random variable for every a, where

$$T_a = \inf \{t : X(t) \leq a\}$$

is the first time the process reaches or goes below a.

Theorem 2: The Generalized IFRA Closure Theorem

If $\{X_i(t)\}$, $i = 1, \ldots, n$, are nonincreasing independent IFRA processes then $\{\phi(\underline{X}(t))\}$ is also IFRA whenever ϕ is nondecreasing.

Proof:

Let $\bar{F}_{i,s}(x) = P\{X_i(s) > x\}$, and suppose first that ϕ is a binary function. Let T denote the first time t that $\phi(\underline{X}(t)) = 0$. Now

$$\begin{aligned} P\{T > \alpha t\} &= P\{\phi(\underline{X}(\alpha t)) = 1\} \quad \text{by monotonicity} \\ &= E[\phi(\underline{X}(\alpha t))] \\ &= r(\bar{F}_{1,\alpha t}, \ldots, \bar{F}_{n,\alpha t}) . \end{aligned} \tag{2}$$

Now

$$\bar{F}_{i,\alpha t}(b) = P\{X_i(\alpha t) > b\}$$

$$= P\{T_{i,b} > \alpha t\}$$

where $T_{i,b}$ denotes the first time that $X_i(t)$ hits or goes below b. Hence from the IFRA hypothesis on $\{X_i(t)\}$ we see that

$$P\{T_{i,b} > \alpha t\} \geq (P\{T_{ib} > t\})^{\alpha}$$

$$= \bar{F}^{\alpha}_{i,t}(b) .$$

Thus

$$\bar{F}_{i,\alpha t}(b) \geq \bar{F}^{\alpha}_{i,t}(b)$$

and so from (2) and Proposition 1

$$P\{T > \alpha t\} \geq r\left(\bar{F}^{\alpha}_{1,t}, \ldots, \bar{F}^{\alpha}_{n,t}\right)$$

$$\geq (r(\bar{F}_{1,t}, \ldots, \bar{F}_{n,t}))^{\alpha} \quad \text{by Theorem 1}$$

$$= (P\{T > t\})^{\alpha}$$

which proves the result when ϕ is binary. For an arbitrary nondecreasing ϕ we can show that the time to go below b is IFRA by using the result in the binary case on the binary function defined by

$$\phi_b(\underline{x}) = \begin{cases} 1 & \text{if} \quad \phi(\underline{x}) > b \\ 0 & \text{if} \quad \phi(\underline{x}) \leq b . \ \blacksquare \end{cases}$$

When ϕ is a binary function we usually say that the system fails at the first t such that $\phi(\underline{X}(t)) = 0$. Thus if each component process has the property that the time it takes to reach or go below any given

level is an IFRA random variable then so is the time to system failure. The following are two examples for which the component processes are IFRA.

Example 1: A Semi-Markov IFRA Process

If $\{X(t), t \geq 0\}$ is a semi-Markov process such that $X(0) \equiv m$, $P_{i,i-1} = 1$, $i \geq 1$, $P_{00} = 1$, and the time in state i is an IFRA random variable, then it follows from the IFRA convolution theorem (Corollary 2) that $\{X(t)\}$ is an IFRA process. This would be a model for a component that gradually went to lower states until it died (reached state 0), spending an IFRA amount of time in each state.

Example 2: A Poisson Shock Model

Suppose the component's level remained constant between times of extreme stress which occurred in accordance with a Poisson process. If at these moments it's level decreased according to a given distribution then the component process would be IFRA. That is, if

$$X(t) = \text{Max}\left\{C - \sum_{i=1}^{N(t)} X_i, 0\right\}$$

where X_i, $i \geq 1$ are independent and identically distributed (i.i.d.) nonnegative random variables that are also independent of the Poisson process $N(t)$, then $\{X(t)\}$ is IFRA. This result was first proven by Esary, Marshall and Proschan [4] who also showed that the same result could be obtained under weaker conditions then the i.i.d. assumption on the X_i.

3. AN NBU CLOSURE THEOREM

We start with some definitions.

Definition:

The distribution $\bar{F}$ with $\bar{F}(0) = 1$ is said to be NBU if $\dfrac{\bar{F}(s + t)}{\bar{F}(s)} \leq \bar{F}(t)$ for all s , $t \geq 0$.

Definition:

The nonincreasing stochastic process $\{X(t), t \geq 0\}$ is said to be NBU if, with probability 1,

$$P\{T_a > s + t \mid X(u), 0 \leq u \leq s\} \leq P\{T_a > t\}$$

for all s , t , $a \geq 0$, where T_a denotes the first time the process hits or goes below a .

Theorem 3:

If the component processes are independent NBU processes then $\{\phi(\underline{X}(t))\}$ is also NBU.

Proof:

Suppose first that ϕ is binary and let T denote the first time the process $\phi(\underline{X}(t))$ hits 0. Now consider

$$P\{T > s + t \mid X_i(u), 0 \leq u \leq s, i = 1, \ldots, n\} =$$
$$E[\phi(\underline{X}(s + t)) \mid X_i(u), 0 \leq u \leq s, i = 1, \ldots, n] .$$

Now it follows from the definition of an NBU process that the conditional distribution of $X_i(s + t)$, given $X_i(u)$, $0 \leq u \leq s$, is stochastically

smaller than the distribution of $X_i(t)$. Hence, from Proposition 1 we see that

$$E[\phi(\underline{X}(s + t)) \mid X_i(u), 0 \leq u \leq s, t = 1, \ldots, n]$$
$$\leq E[\phi(\underline{X}(t))]$$
$$= P\{T > t\}$$

which proves the result when ϕ is binary. As before we can reduce the nonbinary case to the above by defining $\phi_a(\underline{x}) = \begin{cases} 1 & \text{if } \phi(\underline{x}) > a \\ 0 & \text{if } \phi(\underline{x}) \leq a \end{cases}$. ■

Example 3: A Semi-Markov NBU Process

If $\{X(t), t \geq 0\}$ is a semi-Markov process such that $X(0) = m$, $P_{i,i-1} = 1$, $i > 0$, $P_{00} = 0$ and the time in state i is NBU then $\{X(t)\}$ is NBU. This follows from the fact that NBU is preserved under convolution.

Example 4: An NBU Renewal Shock Model

Suppose that $X(0)$ is fixed and that at random time points $X(t)$ is decreased by an i.i.d. nonnegative amount. If the random time points occur in accordance with a renewal process whose interarrival distribution is NBU then it is easy to show that $\{X(t)\}$ is NBU. Formally

$$X(t) = \text{Max}\left\{C - \sum_{i=1}^{N(t)} X_i, 0\right\}$$

where X_i are i.i.d., independent of $\{N(t)\}$ which is a renewal process with an NBU interarrival distribution.

Example 5: An NBU Poisson Shock Model

See Barlow-Proschan [1], p. 160.

4. APPLICATIONS TO STOCHASTIC NETWORKS

The preceding theory can be used in the study of stochastic networks. For instance consider a network and number the arcs. If we let x_i denote the capacity of the $i^{\underline{th}}$ arc then $\phi(\underline{x})$ could be set equal to such quantities of interest as

(i) $\phi_1(\underline{x})$ = max flow from s to t

(ii) $\phi_2(\underline{x}) = \max_P \min_{i\varepsilon P} x_i$

where s and t are two given nodes of the network and P denotes the set of all paths from s to t. Thus $\phi_1(\underline{x})$ equals the maximal flow that can be sent from s to t subject to the arc capacities, and $\phi_2(\underline{x})$, the maximal flow along a single path (previously studied by Barlow [2]). If we interpret x_i as a cost of sending a unit flow along the $i^{\underline{th}}$ arc we could be interested in

$$\phi_3(\underline{x}) = \min_P \sum_{i\varepsilon P} x_i$$

the minimal cost path from s to t. Clearly there are many possible functions ϕ of potential interest, and as it is reasonable to suppose that arc capacities (or costs) could change stochastically with time the results of this paper should be applicable. [For ϕ_3 it may be reasonable to suppose that the x_i increase over time].

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